

ON THE SINGULARITY OF RANDOM BERNOULLI MATRICES — NOVEL INTEGER PARTITIONS AND LOWER BOUND EXPANSIONS

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ABSTRACT. We prove a lower bound expansion on the probability that a random ± 1 matrix is singular, and conjecture that such expansions govern the actual probability of singularity. These expansions are based on naming the most likely, second most likely, and so on, ways that a Bernoulli matrix can be singular; the most likely way is to have a null vector of the form $e_i \pm e_j$, which corresponds to the integer partition 11, with two parts of size 1. The second most likely way is to have a null vector of the form $e_i \pm e_j \pm e_k \pm e_\ell$, which corresponds to the partition 1111. The fifth most likely way corresponds to the partition 21111.

We define and characterize the “novel partitions” which show up in this series. As a family, novel partitions suffice to detect singularity, i.e., any singular Bernoulli matrix has a left null vector whose underlying integer partition is novel. And, with respect to this property, the family of novel partitions is minimal.

We prove that the only novel partitions with six or fewer parts are 11, 1111, 21111, 111111, 221111, 311111, and 322111. We prove that there are fourteen novel partitions having seven parts.

We formulate a conjecture about which partitions are “first place and runners up,” in relation to the Erdős-Littlewood-Offord bound.

We prove some bounds on the interaction between left and right null vectors.

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1. INTRODUCTION

To introduce our problem, we quote verbatim¹ the opening 3 paragraphs of a paper by Kahn, Komlós, and Szemerédi [5]:

“1.1. **The problem.** For M_n a random $n \times n$ ± 1 -matrix (“random” meaning with respect to uniform distribution), set

$$P_n = \Pr(M_n \text{ is singular}).$$

The question considered in this paper is an old and rather notorious one: What is the asymptotic behavior of P_n ?

It seems often to have been conjectured that

$$(1) \quad P_n = (1 + o(1))n^2/2^{n-1},$$

that is, that P_n is essentially the probability that M_n contains two rows or two columns which are equal up to a sign. This conjecture is perhaps best regarded as folklore. It is more or less stated in [6] and is mentioned explicitly, as a standing conjecture, in [9], but has surely been recognized as the probable truth for considerably longer. (It has also been conjectured ([8]) that $P_n/(n^2 2^{-n}) \rightarrow \infty$.)

Of course the guess in (1) may be sharpened, e.g., to

$$(2) \quad P_n - 2^2 \binom{n}{2} \left(\frac{1}{2}\right)^n \sim 2^4 \binom{n}{4} \left(\frac{3}{8}\right)^n,$$

the right-hand side being essentially the probability of having a minimal row or column dependency of length 4.”

The above quoted paper was the first to show that P_n decays exponentially, with an upper bound of $.999^n$. This was later improved by Tao and Vu [10] to $(.958 + o(1))^n$ and again [11] to $(3/4 + o(1))^n$. (See also [12]). Recently Bourgain, Vu, and Wood [3] provided a further improvement to $\left(\frac{1}{\sqrt{2}} + o(1)\right)^n$, which is currently the most accurate bound.

Instead of focusing on upper bounds, we consider *lower* bounds. Our paraphrase of the opening of [5]: (1) says, for a Bernoulli matrix to be singular, the most likely way is to have a left or right null vector of the form $e_i \pm e_j$ for some $1 \leq i, j \leq n$, $i \neq j$, which we say is “of the *template* 11,” and (2) says that the second most likely way to be

¹Apart from correcting a typographical error, and using our own display equation numbering and reference numbering.

singular is to have a left or right null vector of the form $e_i \pm e_j \pm e_k \pm e_\ell$ for distinct indices i, j, k, ℓ , with $1 \leq i, j, k, \ell \leq n$, i.e., of the template 1111.

We use the standard notation for integer partitions: writing $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ implies that the integers λ_i satisfy $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$. When there is no confusion, as in $\lambda = (1, 1, 1, 1)$, we will drop the parentheses and commas and simply write 1111 for the partition. We say that a vector is of the template λ if it is a non-zero multiple of a vector of the form $\lambda_1 e_{i_1} \pm \lambda_2 e_{i_2} \pm \dots \pm \lambda_k e_{i_k}$ for some set of distinct indices $1 \leq i_1, \dots, i_k \leq n$. We define R_λ (resp. L_λ) to be the event that the random matrix has a right (resp. left) null vector of template λ , with

$$(3) \quad D_\lambda := R_\lambda \cup L_\lambda$$

being the event that the random matrix has one or more right or left null vectors of template λ .

The expansion (2) has the form $Q_1(n) (1/2)^n + Q_2(n) (3/8)^n$, where the Q_i are polynomials in n . When one continues the expansion to higher exponential order, two features emerge. First, the templates, corresponding to 11, 1111, \dots , have a rich structure: the real pattern is not simply an even number of 1s, and this first appears in the fifth term, coming from the template 21111. The second feature, also first appearing with the fifth term, is the need to distinguish between the expected number of occurrences of a right or left null vector of template λ , which for $\lambda = 11$ is $2^2 \binom{n}{2} (1/2)^n$, and $\mathbb{P}_n(D_\lambda)$, the probability of one or more such occurrences; see Equations (11) and (12). This is because the exponential decay rate for 21111, which is $(1/4)^n$, is small enough to force consideration of the difference between the *expected number* of occurrences of a right or left null vector of template 11, and the *probability* of one or more such occurrences.

The natural extensions of (2) are our Conjectures 1 and 2, immediately below.

Conjecture 1. *Let S denote the event that the n by n random Bernoulli matrix $M = M_n$ is singular, with $P_n = \mathbb{P}(S) = \mathbb{P}_n(S)$. Then for every $\epsilon > 0$,*

$$\begin{aligned}
(4) \quad \mathbb{P}(S \setminus D_{11}) &= o\left(\left(\frac{3+\epsilon}{8}\right)^n\right) \\
\mathbb{P}(S \setminus (D_{11} \cup D_{1111})) &= o\left(\left(\frac{5+\epsilon}{16}\right)^n\right) \\
\mathbb{P}(S \setminus (D_{11} \cup D_{1111} \cup D_{1^6})) &= o\left(\left(\frac{35+\epsilon}{128}\right)^n\right) \\
\mathbb{P}(S \setminus (D_{11} \cup D_{1111} \cup D_{1^6} \cup D_{1^8})) &= o\left(\left(\frac{1+\epsilon}{4}\right)^n\right) \\
\mathbb{P}(S \setminus (D_{11} \cup D_{1111} \cup D_{1^6} \cup D_{1^8} \cup D_{21111})) &= o\left(\left(\frac{63+\epsilon}{256}\right)^n\right) \\
\mathbb{P}(S \setminus E_6) &= o\left(\left(\frac{15+\epsilon}{64}\right)^n\right) \\
\mathbb{P}(S \setminus E_7) &= o\left(\left(\frac{231+\epsilon}{1024}\right)^n\right) \\
\mathbb{P}(S \setminus E_8) &= o\left(\left(\frac{7+\epsilon}{32}\right)^n\right)
\end{aligned}$$

and so on, where $E_6 = D_{11} \cup D_{1111} \cup D_{1^6} \cup D_{1^8} \cup D_{21111} \cup D_{1^{10}}$, $E_7 = E_6 \cup D_{21^6}$, and $E_8 = E_7 \cup D_{1^{12}}$.

In Section 3 we define what we call *novel* integer partitions. We prove that the set of these is, in a sense, necessary and sufficient for detecting singularities. The precise statements are Theorem 2 (sufficiency), and Theorem 4 (a minimality property which loosely can be called necessity). The denumerability of the set of novel partitions, together with the Erdős, Littlewood, Offord bound (see Proposition 1), allows us to extend Conjecture 1.

Conjecture 2. *For any enumeration $\lambda(1), \lambda(2), \dots$ of the set of novel partitions, for every $r > 0$, there exists $K > 0$ such that*

$$\mathbb{P}\left(S \setminus \bigcup_{i=1}^K D_{\lambda(i)}\right) = o(r^n).$$

Of course, nice enumerations are those for which $K = K(r)$ is minimal, and this corresponds to listing the partitions in nonincreasing order of exponential rate (6). Lemma 1 in the next section *proves* that the first 8 terms on a nice list are $\lambda(1) = 11, \lambda(2) = 1111, \dots, \lambda(5) = 21111, \dots, \lambda(8) = 1^{12}$. Table 3 gives a *plausible* listing, in order, out

to the 59th novel partition, and in this table, the first appearance of a 3 is in $\lambda(25)$.

Section 2 presents an explicit *lower* bound expansion of P_n , whose exponential decay rates are based on the novel integer partitions of Section 3. In Section 4 we derive the polynomial coefficients of our lower bound expansion. In Section 5 we give some bounds on the interaction of potential left and right null vectors, hoping to supply a tool for use in bounding $\mathbb{P}(S \setminus D_{11})$.

2. LOWER BOUND EXPANSIONS

The expansion in (2) can be continued by considering events D_{1111} , that M has a left or right null vector of the form $e_i \pm e_j \pm e_k \pm e_\ell$, with D_{16} , D_{18} , D_{214} , D_{110} , D_{216} , and D_{112} defined similarly. Letting $E_8 = D_{11} \cup D_{14} \cup D_{16} \cup D_{18} \cup D_{214} \cup D_{110} \cup D_{216} \cup D_{112}$, our expansion can be stated as

Theorem 1. *For each n ,*

$$P_n \geq \mathbb{P}(D_{11}) \geq 4 \binom{n}{2} \left(\frac{1}{2}\right)^n - \left(12 \binom{n}{2}^2 - 4 \binom{n}{2}\right) \left(\frac{1}{4}\right)^n.$$

For each n , the event E_8 is a subset of the event that M is singular, hence trivially,

$$P_n \geq \mathbb{P}_n(E_8).$$

Lower and upper bounds on $\mathbb{P}_n(E_8)$ are given by the statement: for all $\epsilon > 0$,

$$\begin{aligned} \mathbb{P}_n(E_8) = & Q_1(n) \left(\frac{1}{2}\right)^n + Q_2(n) \left(\frac{3}{8}\right)^n + Q_3(n) \left(\frac{5}{16}\right)^n + Q_4(n) \left(\frac{35}{128}\right)^n \\ & + Q_5(n) \left(\frac{1}{4}\right)^n + Q_6(n) \left(\frac{63}{256}\right)^n + Q_7(n) \left(\frac{15}{64}\right)^n + Q_8(n) \left(\frac{231}{1024}\right)^n \\ & + o\left(\left(\frac{7+\epsilon}{32}\right)^n\right), \end{aligned}$$

where the polynomial coefficients of the exponentially decaying factors are given by

$$\begin{aligned} Q_1(n) &= 2^2 \binom{n}{2}, \quad Q_2(n) = 2^4 \binom{n}{4} \\ Q_3(n) &= 2^6 \binom{n}{6}, \quad Q_4(n) = 2^8 \binom{n}{8}, \\ Q_5(n) &= 2^5 \binom{5}{1} \binom{n}{5} - 4 \left(2 \binom{n}{2}^2 + 8 \binom{n}{4} + 5 \binom{n}{3} \right), \end{aligned}$$

$$Q_6(n) = 2^{10} \binom{n}{10}, \quad Q_7(n) = 2^7 \binom{7}{1} \binom{n}{7}, \quad Q_8(n) = 2^{12} \binom{n}{12}.$$

Proof. This result follows easily from the combination of Lemmas 1 – 4, given in Sections 3 and 4. \square

3. TEMPLATES, BERNOULLI ORTHOGONAL COMPLEMENTS, AND NOVEL PARTITIONS

When the expansions of Conjecture 1 and Theorem 1 are carried out to high order, an obvious necessary condition for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ to appear is that it be *fairly divisible*, in the sense that for some combination of signs, $0 = \lambda_1 \pm \lambda_2 \pm \dots \pm \lambda_k$. However, this is not sufficient; some fairly divisible partitions, such as 211 and 321, will never appear. We call the partitions that eventually appear *novel*. The definitions below will let us characterize these novel partitions, and, to a limited extent, compute them explicitly.

Definition 1. Integer partition, as a *template* for vectors.

For a given partition λ with k parts, let $V_\lambda \subset \mathbb{N} \times \mathbb{Z}^{k-1}$ denote the set of all vectors formed by reordering the parts of λ , together with all combinations of plus and minus with the requirement that the first coordinate always has a plus.²

If λ has $c(i)$ parts of size i , so that $\text{len}(\lambda) := c(1) + c(2) + \dots = k$, then

$$|V_\lambda| = 2^{k-1} \frac{k!}{c(1)!c(2)! \dots}.$$

Notation: coordinate injection, from \mathbb{R}^k to \mathbb{R}^n .

We often want to pad our vectors of length k with zeros, to get a vector of length n . We say that a k by n matrix C , with all entries 0 or 1, is a *coordinate injection matrix*, if every row has exactly one 1, and no column has more than one 1, and $C_{ij} = C_{i'j'} = 1$ with $i < i'$ implies $j < j'$. (This last requirement is imposed, since our V_λ already accounts for all rearrangements of the parts.) There are $\binom{n}{k}$ such matrices. We speak of vectors of length n , of the form vC for some $v \in V_\lambda$ as *having template λ* .

Definition 2. Templates, used in n dimensions.

We write $V_\lambda^{(n)}$ for the subset of \mathbb{Z}^n of length n vectors with template λ . Note, vectors in $V_\lambda^{(n)}$ may have first coordinate zero, but the first non-zero coordinate must be strictly positive.

²We write $\mathbb{N} := \{1, 2, \dots\}$ for the set of strictly positive integers.

The number of vectors of length n , having template λ , is

$$(5) \quad |V_\lambda^{(n)}| = \binom{n}{k} |V_\lambda| = 2^{k-1} \frac{(n)_k}{c(1)!c(2)!\dots},$$

where we write $(n)_k$ for n falling k .

For an integer partition λ with parts $\lambda_1, \lambda_2, \dots, \lambda_k$, and $X = (\epsilon_1, \dots, \epsilon_k)$ a vector of independent Bernoulli random variables, let $\lambda \cdot X = \lambda_1 \epsilon_1 + \dots + \lambda_k \epsilon_k$ denote the weighted sum, and define

$$(6) \quad r_\lambda := \mathbb{P}(\lambda \cdot X = 0).$$

We can then compute, for example, $r_{11} = 1/2$, $r_{1^{2m}} = \binom{2m}{m}/2^{2m}$, $r_{21111} = 1/4$.

Definition 3. Bernoulli orthogonal complement.

For a vector $v \in \mathbb{Z}^k$,

$$v^{\perp B} = \{x \in \{-1, 1\}^k : v \cdot x = 0\}.$$

This definition can also be applied when $v = \lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \dots \geq \lambda_k > 0$ is an integer partition with k parts, in which case the probability r_λ defined by (6) is given by

$$(7) \quad r_\lambda = \frac{|v^{\perp B}|}{2^k}.$$

Remark 1. Clearly $x \in v^{\perp B}$ iff $-x \in v^{\perp B}$, that is $-v^{\perp B} = v^{\perp B}$. For a partition λ , all v in V_λ have the same size $|v^{\perp B}|$ for their Bernoulli orthogonal complement. Indeed, the various sets $v^{\perp B}$ for $v \in V_\lambda$ are related, by permuting the k coordinates, and applying, for some fixed $I \subset \{2, 3, \dots, k\}$, sign flips to all the coordinates indexed by I . Hence, if $\lambda^{\perp B} \neq \emptyset$, then $\{-1, 1\}^k = \cup_{v \in V_\lambda} v^{\perp B}$.

Definition 4. The matrix $A^{(\lambda)}$ for $\lambda^{\perp B}$.

For an integer partition λ of length k , with $2p = |\lambda^{\perp B}| > 0$, the matrix $A^{(\lambda)}$ for the Bernoulli orthogonal complement of λ is the k by p matrix whose columns are those elements of $\lambda^{\perp B}$ whose first coordinate is $+1$, taken in lexicographic order, with $+1$ preceding -1 .

Example 1. Displaying the Bernoulli orthogonal complement.

When $\lambda = 1111$, we have

$$1111^{\perp B} = \begin{Bmatrix} (+1, +1, -1, -1), \\ (+1, -1, +1, -1), \\ (+1, -1, -1, +1), \\ (-1, -1, +1, +1), \\ (-1, +1, -1, +1), \\ (-1, +1, +1, -1) \end{Bmatrix} = \begin{Bmatrix} + & + & - & -, \\ + & - & + & -, \\ + & - & - & +, \\ - & - & + & +, \\ - & + & - & +, \\ - & + & + & - \end{Bmatrix},$$

where the second representation omits the parentheses and commas for each k -tuple, and also shows only the signs.

Say that λ has length k , and $2p = |\lambda^{\perp B}|$. Showing only those elements of $\lambda^{\perp B}$ that begin with $+$, and transposing, we have a k by p display, to be thought of as an economical representation of the set $\lambda^{\perp B}$; we use this display in Example 4. Treating the same k by p array as a matrix, we have $A^{(\lambda)}$, as defined in Definition 4. For instance,

$$A^{(1111)} = \begin{pmatrix} + & + & + \\ + & - & - \\ - & + & - \\ - & - & + \end{pmatrix}.$$

Definition 5. Equivalence of templates.

For partitions λ, μ with the same number of parts, we say $\lambda \longleftrightarrow \mu$ iff $\exists v \in V_\lambda, w \in V_\mu$, such that $v^{\perp B} = w^{\perp B}$. Clearly, this \longleftrightarrow is an equivalence relation on integer partitions. (Note, $\lambda \longleftrightarrow \mu$ iff $\exists w \in V_\mu$ such that $\lambda^{\perp B} = w^{\perp B}$, that is, we need only apply rearrangement and sign flips to one of λ, μ .)

Example 2. Equivalence is more than just multiples.

Trivially, scalar multiples of any partition are all equivalent to each other. But equivalence involves more. Let $\lambda = 321, \mu = 211$. Then $321 \longleftrightarrow 211$ since

$$\mu^{\perp B} = \lambda^{\perp B} = \left\{ \begin{array}{ccc} + & - & -, \\ - & + & + \end{array} \right\},$$

with no need to apply rearrangements or sign flips. Rearrangement and sign flips may change the Bernoulli complement. For instance,

$$\begin{aligned} V_\mu = & \{(2, 1, 1), (1, 2, 1), (1, 1, 2), (2, 1, -1), (1, 2, -1), (1, 1, -2), \\ & (2, -1, 1), (1, -2, 1), (1, -1, 2), (2, -1, -1), (1, -2, -1), (1, -1, -2)\}. \end{aligned}$$

and with $v = (1, -2, -1) \in V_\mu$, we have

$$v^{\perp B} = \left\{ \begin{array}{ccc} + & + & -, \\ - & - & + \end{array} \right\} \neq \mu^{\perp B}.$$

Example 3. Rearrangements are needed in the definition of equivalence.³

The partitions

$$\mu = 9 \ 7 \ 4 \ 4 \ 3 \ 1$$

³ This example was found by considering partitions of the form $(a + x_1, b + x_1, b, b, a - x_1, b - x_1)$ and $(a + x_2, a - x_2, b + x_2, b, b, b - x_2)$, where $a \geq b, x_1, x_2$ are chosen so that the two b 's must cancel, but are in a different monotonic order in each partition. Here we have taken $a = 6, b = 4, x_1 = 3, x_2 = 1$.

$$\lambda = 7 \ 5 \ 5 \ 4 \ 4 \ 3$$

are such that $\mu^{\perp B} \neq \lambda^{\perp B}$, but for $v = (9, 7, 3, 4, 4, 1) \in V_\mu$, $v^{\perp B} = \lambda^{\perp B}$, hence $\mu \longleftrightarrow \lambda$.

Definition 6. Reduction of templates.

For any partitions μ, λ with μ having m parts and λ having k parts, $m \geq k > 0$, we say that $\mu \longrightarrow \lambda$ (read μ reduces to λ or μ implies λ) iff either

$$(8) \quad k = m \text{ and } \exists v \in V_\lambda, \mu^{\perp B} \subset v^{\perp B},$$

or else

$$(9) \quad \exists I \subset \{1, \dots, m\}, v \in V_\lambda, \text{Proj}_I \mu^{\perp B} \subset v^{\perp B}.$$

Clearly, the relation \longrightarrow is transitive. Our use of the subset symbol \subset includes equality. We note that $(\lambda \longrightarrow \mu \text{ and } \mu \longrightarrow \lambda)$ iff $\lambda \longleftrightarrow \mu$, so that definitions 5 and 6 are compatible.

Remark 2. Definition 6 is set up so that it is obvious that if $\mu \longrightarrow \lambda$, and $w \in V_\mu^{(n)}$, and M is an n by n Bernoulli matrix with $wM = 0$, then there exists $v \in V_\lambda^{(n)}$ with $vM = 0$.

Definition 7. Strict reduction.

We define a relation of strict reduction, $\mu \not\longrightarrow \lambda$ (read μ strictly reduces to λ) iff $\mu \longrightarrow \lambda$ and not $\lambda \longrightarrow \mu$. Hence, $\mu \not\longrightarrow \lambda$ iff (8) with proper subset containment of the Bernoulli complements, or (9) holds. Clearly, the relation $\not\longrightarrow$ is transitive and irreflexive.

Example 4. Strict reduction using (8).

$\mu := 332211 \not\longrightarrow \lambda := 221111$.

$$A^{(332211)} = \begin{pmatrix} + & + & + & + & + \\ + & - & - & - & - \\ - & + & + & - & - \\ - & - & - & + & + \\ - & + & - & + & - \\ - & - & + & - & + \end{pmatrix}, A^{(221111)} = \begin{pmatrix} + & + & + & + & + & + & + \\ + & - & - & - & - & - & - \\ - & + & + & + & - & - & - \\ - & + & - & - & + & + & - \\ - & - & + & - & + & - & + \\ - & - & - & + & - & + & + \end{pmatrix}$$

Upon visual inspection, it is easily seen that each column of $A^{(332211)}$ appears as a column in $A^{(221111)}$, which shows that $332211^{\perp B} \subset 221111^{\perp B}$.

Example 5. Strict reduction using (9).

The partition 211 reduces to the partition 11.

$$A^{(211)} = \begin{pmatrix} + \\ - \\ - \end{pmatrix}, \quad A^{(11)} = \begin{pmatrix} + \\ - \end{pmatrix}.$$

Take $I = \{1, 2\}$, so that projection onto I “forgets” the third coordinate in $211^{\perp B}$. We then have $\text{Proj}_I 211^{\perp B} = 11^{\perp B}$, and $211 \not\rightarrow 11$.

Example 6. The consequence of not implying 11.

If λ does not imply 11, then every two rows of $A^{(\lambda)}$ are linearly independent. Thus, for every $i \neq j$, both $\lambda_i + \lambda_j$ and $\lambda_i - \lambda_j$ are expressible as a plus-minus combination of the remaining parts. (The proof of Proposition 5 uses this.)

There is a natural description of this principle in terms of coin weighing problems (see for example [1]). You have k coins of various positive integer weights. Not implying 11 means that if an adversary selects any two coins and places them on the same or opposite sides of a balance scale, you can place *all* of the remaining coins on the scale so that it balances.

We now come to the definition that effectively governs explicit expansions such as those in Conjecture 1 and Theorem 1.

Definition 8. Novel partitions.

We call an integer partition λ a novel partition if and only if there does not exist any other partition λ' with $\lambda \not\rightarrow \lambda'$, and among all partitions equivalent to λ , in the sense of Definition 5, λ is lexicographically first.

Theorem 2 (Sufficiency of the set of novel partitions). *The set of all novel partitions is sufficient, acting as possible left null vectors, to detect singularity for Bernoulli matrices M . That is, if such a matrix is singular, say of size n by n , then there exists a novel partition λ with $\text{len}(\lambda) \leq n$, and $v \in V_\lambda^{(n)}$ with $vM = 0$.*

Proof. If M is singular, then there is a nonzero vector $w \in \mathbb{Z}^n$ with $wM = 0$. Taking absolute values of the coordinates, deleting zeros if they occur, and listing in nonincreasing order yields an integer partition λ , and $w \in V_\lambda^{(n)}$. If λ is novel, we are done. If λ is not novel, then it must reduce to a novel partition μ , and then Remark 2 applies. \square

Theorem 3. Intrinsic characterization of novel partitions.

An integer partition λ with k parts is novel iff the matrix $A^{(\lambda)}$, specified in Definition 4, has rank $k - 1$, and $\gcd(\lambda) = 1$.

Proof. Let $A \equiv A^{(\lambda)}$, with rows r_1, \dots, r_k . To prove the only if direction, suppose $\text{rank } A < k - 1$. Then there exists $j < k$, and integers $c_1, \dots, c_j \neq 0$, π_1, \dots, π_j distinct elements of $\{1, \dots, k\}$, such

that $c_1 r_{\pi_1} + \dots c_j r_{\pi_j} = 0$. Letting $v = (c_1, \dots, c_j)$, we have $v \in V_\mu$, $\text{len}(\mu) = j < k$, and $\lambda \not\rightarrow \mu$, so that λ is not novel.

If $\gcd(\lambda) > 1$, then $\mu = \frac{1}{\gcd(\lambda)}\lambda$, $\mu \longleftrightarrow \lambda$, μ is earlier in lexicographic order, so λ is not novel.

In the other direction, suppose $\text{rank } A = k - 1$, $\gcd(\lambda) = 1$, but assume λ is not novel. Then either

- (1) $\lambda \not\rightarrow \mu$, $\text{len}(\mu) < \text{len}(\lambda)$, but then there exists $v \in V_\mu^{(k)}$, with $k - 1$ or fewer nonzero components such that $vA = 0$, which implies $\text{rank } A < k - 1$;
- (2) $\lambda \not\rightarrow \mu$, $\text{len}(\mu) = \text{len}(\lambda)$, $\lambda^{\perp B} \subsetneq \mu^{\perp B}$. Then $A^{(\mu)}$ and $A^{(\lambda)}$ both have rank $k - 1$, with $\mu A^{(\mu)} = 0$ and $\lambda A^{(\lambda)} = 0$. By the inclusion, we also have $\mu A^{(\lambda)} = 0$. But then if we consider the vector $v = \mu_1 \lambda - \lambda_1 \mu$ of length k with first coordinate 0, v has at most $k - 1$ nonzero entries, and $v A^{(\lambda)} = 0$. Since we assumed $A^{(\lambda)}$ has rank $k - 1$, we conclude $v = 0$.

□

Corollary 1. *An integer partition λ is either*

- (1) *novel (or a multiple of a novel),*
- (2) *implies a novel partition μ of strictly smaller length, or*
- (3) *is not fairly divisible, i.e., $\lambda^{\perp B} = \emptyset$.*

The only part that is not trivial is (2). We already showed that partitions like 332211 can strictly reduce to a partition of the *same* length, but without Theorem 3 it is not a priori obvious that there will always be a strict reduction in the length of the partition.

Theorem 4 (Minimality of the set of novel partitions). *The family of novel partitions is minimal, in the sense that if any single one of the λ s in that set is removed, then the family, acting as possible left null vectors, does not detect all singularities.*

Proof. Fix a novel partition λ , of length k . We will construct a singular Bernoulli matrix M with the property that every left null vector of M , having integer coordinates with greatest common divisor 1, is of template λ .

Let $2p = |\lambda^{\perp B}|$; we can write $\lambda^{\perp B} = \{x_1, x_2, \dots, x_{2p}\}$, where $x_i \in \{-1, 1\}^k$, $x_i \neq x_j$, $i \neq j$. As in Example 1, let $A \equiv A^{(\lambda)}$ denote the k by p matrix of rank $k - 1$ with columns given by x_1, x_2, \dots, x_p , the vectors with first entry positive.

- (1) If $p \leq k$, then add $k - p$ columns which are duplicates of column p , and call this square matrix M . Note, since A had rank $k - 1$,

either A was k by k and $M = A$, or else we added exactly one column. In either subcase, M is k by k of rank $k - 1$.

- (2) If $p > k$, since the rank of A is $k - 1$, the first $k - 1$ rows, denoted r_1, r_2, \dots, r_{k-1} , form an independent set in \mathbb{R}^p . The existence of an independent set of p vectors in \mathbb{R}^p , whose entries consist of plus and minus 1, is guaranteed by the existence of nonsingular Bernoulli matrices of all sizes; denote such a set as $\{s_1, \dots, s_p\}$. By the basis extension theorem, re-indexing the s_i as needed, there is an independent set of the form $B = \{r_1, r_2, \dots, r_{k-1}, s_k, s_{k+1}, \dots, s_p\}$. Replacing s_k with r_k , the rows $r_1, r_2, \dots, r_{k-1}, r_k, s_{k+1}, \dots, s_p$ form a p by p Bernoulli matrix M with rank $p - 1$.

In either case, we have a square matrix M of corank 1. Suppose we have a left null vector w with integer coordinates. In case (1), this implies $\lambda^{\perp B} \subset w^{\perp B}$, and since λ was novel, this implies $w \in V_\lambda$. In case (2), write $w = (w_1, \dots, w_k, \dots, w_p)$. The condition $wM = 0$ says that, in row space, $0 = w_1 r_1 + \dots + w_k r_k + w_{k+1} s_{k+1} + \dots + w_p s_p$, and the independence of the set B now implies that $0 = w_1 r_1 + \dots + w_k r_k$ and $w_{k+1} = \dots = w_p = 0$. With $v = (w_1, \dots, w_k)$, we have $\lambda^{\perp B} \subset v^{\perp B}$, and since λ was novel, this implies $v \in V_\lambda$ and $w \in V_\lambda^{(p)}$. \square

Remark 3. *In Theorem 4, we specified testing for null vectors on one particular side, since in the case $n = 4$ any instance of singularity detected by a null vector of template 1111 on one side implies that there is a null vector of template 11 on the other side. Without having specified a side, one could say that 1111 is not necessary to detect singularity of 4 by 4 Bernoulli matrices; the template 11 by itself suffices.*

We have just defined and characterized novel partitions, which form the foundation for the expansion in Theorem 1. The next set of theorems bounds the exponential decay from each term.

Proposition 1 (Erdős, Littlewood, Offord [4]). *Let x_1, x_2, \dots be real numbers, $|x_i| \geq 1$, and $\epsilon_1, \epsilon_2, \dots$ be $+1$ or -1 . Then the number of sums of the form $\sum_{i=1}^k x_i \epsilon_i$ which fall into an arbitrary open interval I of length 2 does not exceed $\binom{k}{\lfloor k/2 \rfloor}$.*

Taking $I = (-1, 1)$, an immediate consequence is that for any integer partition λ with k parts,

$$(10) \quad 2^k r_\lambda = |\lambda^{\perp B}| \leq \binom{k}{\lfloor \frac{k}{2} \rfloor},$$

and in case $k = 2m$ is even, the novel partition $\lambda = 1^{2m}$ achieves equality with this upper bound.

A related theorem of Erdős [4] expands Proposition 1 by widening the target interval.

Proposition 2 (Erdős [4]). *Let r be any integer, the x_i real, $|x_i| \geq 1$. Then the number of sums $\sum_{i=1}^k \epsilon_i x_i$ which fall into the interior of any interval of length $2r$ is not greater than the sum of the r greatest binomial coefficients belonging to k .*

This proposition was proved by showing that the size of the union of r disjoint antichains in $\{-1, 1\}^k$ is at most the sum of the r largest binomial coefficients for k ; see [12], Proposition 7.7, and [2], Section 3, Exercise 7. As a corollary of this, we get

Theorem 5. *Suppose λ is an integer partition with k parts, not all equal. Then $2^k r_\lambda = |\lambda^{\perp B}|$ is at most the sum of the largest four binomial coefficients of $k - 2$. Hence, for $k \geq 2$ and even, $\lambda = 1^k$ has $|\lambda^{\perp B}| > |\mu^{\perp B}|$ for any partition μ with k parts, not all equal, and for $k \geq 5$ and odd, $\lambda = 2 \cdot 1^{k-1}$ has $|\lambda^{\perp B}| \geq |\mu^{\perp B}|$, for any partition μ with k parts.*

Proof. Fix i, j such that $\lambda_i \neq \lambda_j$. Partition the set $\lambda^{\perp B}$ into four (possibly empty) subsets

$$\begin{aligned} A &= \{x \in \lambda^{\perp B} : x_i = 1, x_j = 1\}, \\ B &= \{x \in \lambda^{\perp B} : x_i = -1, x_j = -1\}, \\ C &= \{x \in \lambda^{\perp B} : x_i = 1, x_j = -1\}, \\ D &= \{x \in \lambda^{\perp B} : x_i = -1, x_j = 1\}. \end{aligned}$$

These are disjoint antichains, since they specify four distinct target values for the sums $\sum x_\ell \lambda_\ell$, where the sum is over the $k - 2$ indices other than i, j , and each λ_ℓ is strictly positive. Projecting out the two coordinates indexed by i and j , we get 4 disjoint antichains in $\{-1, +1\}^{k-2}$. \square

Conjecture 3. Runners-up in Erdős-Littlewood-Offord.

For all partitions λ with exactly k parts, with greatest common divisor 1, if $k \geq 4$ is even, the second largest probability r_λ is achieved, uniquely, by $2^2 1^{k-2}$, while if $k \geq 7$ is odd, the largest probability is achieved by $2 \cdot 1^{k-1}$ (already proved, as part of Theorem 5), and the second largest is achieved, uniquely, by $2^3 1^{k-3}$.

We note that for $k \geq 5$ odd, it is trivial to check that $2 \cdot 1^{k-1}$ strictly beats $2^3 1^{k-3}$, and with $k = 5$, $|22211^{\perp B}| = |32111^{\perp B}| = 6$.

Proposition 3. *There are no novel partitions of size three.*

Proof. By Theorem 3, any novel partition λ with three parts must have rank $A^{(\lambda)} = 2$. Since 111 is not a valid template, the parts in λ are not all equal, and so by Theorem 5, $|\lambda^{\perp B}| \leq 2$, which means that $A^{(\lambda)}$ has at most 1 column, and hence rank at most 1. \square

Proposition 4. *The only novel partition of size 4 is 1111.*

Proof. By Theorem 3, any novel partition λ with four parts must have rank $A^{(\lambda)} = 3$. By Theorem 5, any novel partition λ with four parts, not all equal, has $|\lambda^{\perp B}| \leq 4$, which means that $A^{(\lambda)}$ has at most 2 columns, and hence rank at most 2. If all parts of the partition are equal, then the requirement $\gcd(\lambda) = 1$ forces $\lambda = 1111$. This is indeed novel, with $A^{(1111)}$ given in Example 1. \square

Proposition 5. *The only novel partition of size 5 is 21111.*

Proof. Without loss of generality, assume $\lambda = (a, b, c, d, e)$, where $a \geq b \geq c \geq d \geq e > 0$. As described in Example 6, in order to avoid implying 11, every pair of parts in λ , when added or subtracted, must be a signed combination of the others, e.g.,

$$\begin{aligned} a + b &= \pm c \pm d \pm e \\ b + c &= \pm a \pm d \pm e. \end{aligned}$$

Let us look at the first equation. If any of the signs are negative, then monotonicity is necessarily broken. Thus, any novel partition of length five must have $a + b = c + d + e$. Similarly, we can look at $b + c = \pm a \pm d \pm e$, and by a monotonicity argument we conclude that the only viable form is $b + c = a \pm d \pm e$. We will look at each of these four cases separately.

(1)

$$\begin{aligned} a + b &= c + d + e \\ b + c &= a + d + e \end{aligned}$$

can be refined (by adding or subtracting one from the other) to $b = d + e$ and $a = c$. By monotonicity this means that $a = b = c$, and hence our partition would be of the form $(d + e, d + e, d + e, d, e)$. However, we must have a solution to $d - e = \pm(d + e) \pm (d + e) \pm (d + e)$, which would imply that $e = 0$.

(2)

$$\begin{aligned} a + b &= c + d + e \\ b + c &= a + d - e \end{aligned}$$

can be refined similarly to $b = d$ and $a = c + e$, which yields partitions of the form $(c + e, c, c, c, e)$. We must have a solution

to $c + 2e = \pm c \pm c \pm c$, which, to avoid implying $e \leq 0$, implies $e = c$, and our template reduces to a multiple of 21111.

(3)

$$\begin{aligned} a + b &= c + d + e \\ b + c &= a - d + e \end{aligned}$$

can be refined to $b = e$, $a = c + d$, hence a multiple of 21111.

(4)

$$\begin{aligned} a + b &= c + d + e \\ b + c &= a - d - e \end{aligned}$$

forces $b = 0$.

□

Proposition 6. *The only novel partitions of length 6 are 111111, 221111, 311111, 322111. The only novel partitions of length 7 are*

$$\begin{aligned} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 2 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 1 & 1 & 1 & 1 \\ 4 & 3 & 3 & 2 & 2 & 1 & 1 \\ 5 & 2 & 2 & 2 & 1 & 1 & 1 \\ 5 & 3 & 3 & 2 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 2 & 1 & 1 \end{aligned}$$

Proof. The same technique that was used in Proposition 5 can be continued for novel partitions of length 6, 7, etc., eliminating cases that imply 11. `Mathematica` [7] code was written to list all cases and reduce them. For the sake of economy in running time, we only considered the requirement that all four of $\lambda_1 \pm \lambda_2$ and $\lambda_{k-1} \pm \lambda_k$ be expressible plus-minus combination of the other $k - 2$ parts. When the reduction yields a space of dimension greater than one, the result may be viewed as what we call a meta-template, e.g., $(a + b, a + b, b, b, a, a)$. The list of meta-templates includes all novel partitions and possibly others that

i	$\lambda(i)$	$len(\lambda)$	r_λ	$256 \cdot r_\lambda$
1	11	2	1/2	128.
2	1111	4	3/8	96.
3	111111	6	5/16	80.
4	11111111	8	35/128	70.
5	21111	5	1/4	64.
6	111111111	10	63/256	63.
7	2111111	7	15/64	60.
8	11111111111	12	231/1024	57.75
9	221111	6	7/32	56.
10	21111111	9	7/32	56.
11	111111111111	14	429/2048	53.625
12	2111111111	11	105/512	52.5
13	22111111	8	13/64	52.
14	11111111111111	16	6435/32768	50.2734
15	211111111111	13	99/512	49.5
16	2211111111	10	49/256	49.
17	2221111	7	3/16	48.
18	1111111111111111	18	12155/65536	47.4805
19	21111111111111	15	3003/16384	46.9219
20	221111111111	12	93/512	46.5
21	22211111	9	23/128	46.
22	111111111111111111	20	46189/262144	45.1064
23	2111111111111111	17	715/4096	44.6875
24	22111111111111	14	1419/8192	44.3438
25	3211111	7	11/64	44.
26	22221111	8	11/64	44.
27	2221111111	11	11/64	44.
28	11111111111111111111	22	88179/524288	43.0562
29	211111111111111111	19	21879/131072	42.7324
30	2211111111111111	16	2717/16384	42.4531
31	22211111111111	13	675/4096	42.1875
32	31111111	8	21/128	42.
33	33111111	8	21/128	42.
34	3211111111	9	21/128	42.
35	3111111111	10	21/128	42.
36	2222111111	10	21/128	42.
37	1111111111111111111111	24	676039/4194304	41.2621
38	21111111111111111111	21	20995/131072	41.0059
39	221111111111111111	18	10439/65536	40.7773
40	321111111111	11	81/512	40.5
41	222211111111	12	323/2048	40.375
42	31111111111111	14	1287/8192	40.2188
43	311111	6	5/32	40.
44	322111	6	5/32	40.
45	3222111	7	5/32	40.
46	32211111	8	5/32	40.
47	222221111	9	5/32	40.
48	111111111111111111111111	26	1300075/8388608	39.6751
49	2111111111111111111111	23	323323/2097152	39.4681
50	22111111111111111111	20	20111/131072	39.2793
51	32111111111111	13	627/4096	39.1875
52	3111111111111111	16	5005/32768	39.1016
53	3221111111	10	39/256	39.
54	3311111111	10	39/256	39.
55	22221111111111	14	623/4096	38.9375
56	222221111111	11	155/1024	38.75
57	11111111111111111111111111	28	5014575/33554432	38.2582
58	211111111111111111111111	25	156009/1048576	38.0881
59	3222111111	9	19/128	38.

TABLE 1. Novel partitions sorted by r_λ . Conjectured to be complete with respect to $r_\lambda \geq 38/256$.

are not novel. For $k = 6$, the following candidates were returned:

$$111111, 221111, 311111, 322111, 332211, 433211, 533221.$$

We showed in Example 4 that $332211 \not\rightarrow 221111$, and the ranks of $A^{(433211)}$ and $A^{(533221)}$ are 4, whereas the others have rank 5.

For $k = 7$, a list of 14 templates was found; all of which turned out to be novel. Also, for $k = 7$, 12 meta-templates were found; by hand inspection, 11 were easily shown to violate the monotonicity requirement that $\lambda_1 \geq \lambda_2 \geq \dots$. The remaining meta-template, $(a, b, a - b, d, e, h, h - d - e)$ is seen, by hand, to imply 11, since after the initial refinement to the form above one can apply the same technique *again* to the smallest two parts and reduce each case to either a monotonicity or positivity violation. \square

Lemma 1. *In order of decreasing r_λ , the first eight novel partitions λ are $11, 1111, 1^6, 1^8, 21111, 1^{10}, 21^6, 1^{12}$. For novel partitions other than these eight, writing k for the number of parts*

$$\begin{aligned} k = 6, 7 : \quad r_\lambda &\leq \frac{60}{256}, \\ k = 8, 9 : \quad r_\lambda &\leq \frac{56}{256}, \\ k = 10, 11 : \quad r_\lambda &\leq \frac{52.5}{256}, \\ \text{not } 1^{14}, 1^{16}, \text{ and } k \geq 12 : \quad r_\lambda &\leq \frac{49.5}{256}. \end{aligned}$$

Hence, aside from the first eight novel partitions, all other novel partitions have $r_\lambda \leq 56/256$. Observe that $\lambda = 1^{14}$ has $r_\lambda = 53.625/256$ and $\lambda = 1^{16}$ has $r_\lambda = 50.2734375/256$.

Proof. This follows immediately from Theorem 5. For example, when $k = 6$ the four largest binomial coefficients of $k - 2$ appear on the left side below, and

$$\binom{4}{2} + \binom{4}{1} + \binom{4}{3} + \binom{4}{0} = 15,$$

with $15/2^6 = 60/256$ giving our upper bound for $k = 6$. \square

Conjecture 4. *In order of decreasing r_λ , the novel partitions with $r_\lambda \geq 38/256$ are precisely those given in Table 3.*

Example 7. The shortest novel arithmetic progression.

The partition $\lambda = (8, 7, 6, 5, 4, 3, 2, 1)$ is novel. It has $|\lambda^{\perp B}| = 14$, so

λ	$ \lambda^{\perp B} $	λ	$ \lambda^{\perp B} $	λ	$ \lambda^{\perp B} $
11111111	70	54433221	22	65522211	16
22111111	52	53332211	22	65533211	16
22221111	44	65322211	22	65544332	16
33111111	42	64322111	22	76433221	16
31111111	42	64432111	22	76533211	16
32211111	40	63222111	22	76543221	16
33221111	36	55422211	20	76544321	16
32222111	36	55433211	20	75543211	16
33222211	34	65332111	20	87433221	16
43211111	32	65432211	20	86433211	16
42111111	32	64331111	20	86543211	16
42221111	32	64332211	20	85432211	16
33311111	30	63321111	20	85542211	16
33322111	30	63322211	20	84332211	16
44221111	30	44333111	18	55443331	14
43222111	30	55333111	18	54411111	14
43321111	30	55443221	18	53332222	14
43322211	28	54433111	18	51111111	14
33322221	26	54433322	18	76433111	14
44322111	26	65332221	18	76522211	14
44332211	26	65422111	18	76544211	14
43332111	26	65433111	18	76554331	14
43332221	26	65433221	18	75443322	14
54222111	26	65443211	18	75522111	14
53221111	26	65443321	18	74333222	14
53322111	26	65543221	18	74431111	14
53331111	26	64421111	18	73331111	14
52222111	26	64433211	18	72222111	14
54321111	24	62221111	18	87533211	14
54322211	24	76332221	18	87543221	14
54332111	24	76432211	18	87654321	14
53222211	24	75332211	18	86533111	14
53311111	24	75432111	18	85532111	14
52211111	24	75433211	18	83332111	14
44311111	22	75442211	18	98543221	14
44333221	22	75533111	18	97543211	14
55322111	22	74322211	18	97644211	14
54332221	22	74422111	18	96542211	14
54333211	22	74432211	18	95532211	14
54422111	22	73322111	18	94432211	14
54432211	22	73332211	18		

TABLE 2. Novel partitions of length 8, conjectured to be the complete list.

$A^{(\lambda)}$ is an 8 by 7 matrix of rank 7. Examination of the $21 = 1 + 0 + 1 + 1 + 4 + 14$ novel partitions of lengths 2, 3, 4, 5, 6, 7 in Propositions 4 – 6 shows that this λ is the shortest novel partition which is also an arithmetic progression.

Conjecture 5. *There are exactly 122 novel partitions of length $k = 8$.*

The list of 122 is given in Table 2. Our evidence in favor of this conjecture is that these 122, and no others, were found by a random survey, using `Mathematica`, of 420 million singular n by n matrices M , for $n = 8$. Of course, this is not a proof. For an exhaustive search, to guarantee that all novel partitions of length 8 have been found, one might observe that, with respect to the integer partitions underlying potential right and left null vectors, M can be taken to have first row and first column all +1, so that it would suffice to examine 2^{49} matrices M .

Remark 4. *The `Mathematica` command `NullSpace` applied to a singular n by n Bernoulli matrix M returns a list of length n vectors that forms a basis for the null space of M . Aside from the sign requirement in the first nonzero entry, these vectors have always been of the form $v \in V_{\lambda}^{(n)}$ for some novel partition λ . One would like to prove a result about this, but since the basis returned by a generic null space algorithm is not unique, and hence implementation dependent, we will not pursue this idea further.*

4. POLYNOMIAL COEFFICIENTS ARISING FROM INCLUSION-EXCLUSION

For events $\{A_{\alpha}\}_{\alpha \in I}$, for a finite index set I , and $A = \cup_{\alpha \in I} A_{\alpha}$, the inclusion-exclusion formula states that

$$(11) \quad \mathbb{P}(A) = \sum_{\alpha \in I} \mathbb{P}(A_{\alpha}) - \sum_{\{\alpha, \beta\} \subset I, \alpha \neq \beta} \mathbb{P}(A_{\alpha} \cap A_{\beta}) + \sum \mathbb{P}(A_{\alpha} \cap A_{\beta} \cap A_{\gamma}) \\ + \cdots + (-1)^{|I|-1} \mathbb{P}(\cap_{\alpha \in I} A_{\alpha}).$$

With $W = \sum_{\alpha \in I} 1(A_{\alpha})$, a sum of indicators of the events, the formula above may be expressed as

$$(12) \quad \mathbb{P}(A) = \mathbb{E} W - \mathbb{E} \binom{W}{2} + \mathbb{E} \binom{W}{3} - \cdots + (-1)^{|I|-1} \mathbb{E} \binom{W}{|I|}.$$

The Bonferroni inequalities state that for events $\{A_{\alpha}\}_{\alpha \in I}$, for a finite index set I , and $A = \cup_{\alpha \in I} A_{\alpha}$,

$$\begin{aligned}
\mathbb{P}(A) &\leq \sum_{\alpha \in I} \mathbb{P}(A_\alpha) \\
(13) \quad \mathbb{P}(A) &\geq \sum_{\alpha \in I} \mathbb{P}(A_\alpha) - \sum_{\{\alpha, \beta\} \subset I, \alpha \neq \beta} \mathbb{P}(A_\alpha \cap A_\beta) \\
&\quad \dots \dots \dots
\end{aligned}$$

Equation 13 is a lowerbound, with the \dots representing higher order bounds. A variation of (13), with $B = \cup_{\beta \in I'} A_\beta$,

$$(14) \quad \mathbb{P}(A \setminus B) \geq \sum_{\alpha \in I} \mathbb{P}(A_\alpha) - \sum_{\{\alpha, \beta\} \subset I, \alpha \neq \beta} \mathbb{P}(A_\alpha \cap A_\beta) - \sum_{\alpha \in I, \beta \in I'} \mathbb{P}(A_\alpha \cap A_\beta),$$

is proved similarly.

We take $I = \binom{[n]}{2} \times \{-, +\} \times \{L, R\}$,⁴ so that $\alpha \in I$ specifies a set of two distinct indices along with sign and direction bits. The event A_α corresponds to the occurrence of a null vector of the form α . For example, $\alpha = (\{2, 5\}, -, R) \in I$, and A_α is the event that $e_2 - e_5$ is a right null vector.

Proposition 7. *For $W = \sum_{\alpha \in I} 1(A_\alpha)$ with I and the A_α as above, so that $D_{11} = \{W > 0\}$,*

$$\begin{aligned}
\mathbb{E} W &= 4 \binom{n}{2} \left(\frac{1}{2}\right)^n, \\
\mathbb{E} \binom{W}{2} &= \left(12 \binom{n}{2}^2 - 4 \binom{n}{2}\right) \left(\frac{1}{4}\right)^n, \\
\mathbb{E} \binom{W}{3} &= 2^2 \binom{n}{3} \left(\frac{1}{4}\right)^n + \\
&\quad + 2^{3-3n} \left(\frac{13}{3} \binom{n}{2}^3 - 4 \binom{n}{2}^2 - \frac{2}{3} \binom{n}{2} - \frac{1}{3} \binom{n}{3} \binom{3}{2}^3 - \binom{n}{2} \binom{n-2}{2}\right) \\
&= 4 \binom{n}{3} \left(\frac{1}{4}\right)^n + O(n^6 2^{-3n}).
\end{aligned}$$

Proof. Let $t = 2 \binom{n}{2} 2^{-n}$. Clearly,

$$\mathbb{E} W = \sum_{\alpha \in I} P(e_i = \pm e_j) = |I| 2^{-n} = 4 \binom{n}{2} 2^{-n} = 2t.$$

⁴The notation used here is: $[n]$ is the set $\{1, 2, \dots, n\}$, and for a set T and nonnegative integer k , $\binom{T}{k}$ is the set of all k -subsets of T .

Let I_R (resp. I_L) be the set of $\alpha \in I$ with last coordinate R (resp. L). Let $B_\alpha = 1(A_\alpha)$ be the indicator random variable, for $\alpha \in I$. Then

$$\mathbb{E} \binom{W}{2} = 2 \sum_{\{\alpha, \beta\} \subset I_R, \alpha \neq \beta} \mathbb{E} B_\alpha B_\beta + \sum_{\alpha \in I_R, \beta \in I_L} \mathbb{E} B_\alpha B_\beta =: G_1 + G_2.$$

The first sum corresponds to two null vectors both on the right; with a factor of 2 we get the contribution G_1 for both null vectors on the same side, either right or left. The second contribution G_2 corresponds to two null vectors on opposite sides. We have

$$(15) \quad G_1 = t^2 - 2^2 \binom{n}{2} 2^{-2n} = t^2 - t 2^{1-n},$$

$$(16) \quad G_2 = 2 \binom{n}{2} 2^{-n} \times 2 \binom{n}{2} 2^{-n} \times 2 = 2t^2.$$

The first equation (15) is found by considering all pairs on one side, and then taking away all pairs that share two rows along with any plus or minus combination. The second equation (16) has a “boost” factor of 2, since for α, β on opposite sides, both of the template 11, we have $\mathbb{P}(A_\alpha | A_\beta) = 2\mathbb{P}(A_\alpha)$.

Finally, we have

$$6 \mathbb{E} \binom{W}{3} = \sum_{(\alpha, \beta, \gamma)} 1(\alpha, \beta, \gamma \text{ distinct}) \mathbb{E} B_\alpha B_\beta B_\gamma =: F_1 + F_2$$

Here F_1 denotes the sum over all events where α, β, γ appear on the same side, and F_2 denotes the sum over events where two appear on one side, and one on the other. By choosing a side (left or right), we have for some t_2, t_3, t_4 functions of n ,

$$\frac{F_1}{2} = t^3 - t_2 - t_3 - t_4 + 2^2 \binom{3}{2} \binom{n}{3} \left(\frac{1}{2}\right)^{2n}.$$

The t^3 considers all triplets, and the t_i considers the triplets that are supported on i rows, $i = 2, 3, 4$, that need to be excepted. We have

$$t_2 = \binom{n}{2} 2^3 2^{-3n},$$

which chooses any two rows and all sign combinations. When three rows are supported, there are precisely $2^3 \binom{3}{1}^3$ combinations, but events of the form $\{e_i \pm e_j, e_i \pm e_k, e_j \pm e_k \text{ are null vectors}\}$ are sometimes valid. When they are valid, they have a probability of $(1/2)^{2n}$, hence

excepting *all* events involving three rows we have

$$t_3 = \binom{n}{3} 2^3 \binom{3}{1}^3 2^{-3n},$$

and the term at the end adds back in the valid combinations supporting three rows. These events are of the form $\{e_i \pm e_j \text{ and } e_i \pm e_k \text{ are null vectors}\}$, which imply one of $e_j + e_k$ or $e_j - e_k$ is a null vector as well. Finally, when four rows are supported, the exceptional cases are those in which two of α, β, γ share two rows, and one does not share any, thus

$$t_4 = \binom{3}{1} \binom{n}{2} \binom{n-2}{2} 2^3 2^{-3n}.$$

For F_2 , there are two choices for which side the solo index appears, and then three choices for which of α, β, γ is this solo index. We have

$$\frac{F_2}{6} = 4t^3 - \binom{n}{2} 2^2 \times \binom{n}{2} 2 \times 2^{2-3n}.$$

The factor of four comes from $\mathbb{E} B_\alpha B_\beta B_\gamma = 4\mathbb{E} B_\alpha \mathbb{E} B_\beta \mathbb{E} B_\gamma$, which is a boost from conditioning on an opposite side. The exceptional cases are those where the support of the non-solo pair lie on the same two rows, and includes all sign combinations. The solo index can be anything, and gets a conditional boost from being on the other side. \square

Proposition 8. *Recall (3), and that R_λ (resp. L_λ) denotes the event that there is a right (resp. left) null vector of template λ . We have*

$$\begin{aligned} \mathbb{P}(R_{11} \setminus L_{11}) &\geq 2 \binom{n}{2} \left(\frac{1}{2}\right)^n - \left(12 \binom{n}{2}^2 - 4 \binom{n}{2}\right) \left(\frac{1}{4}\right)^n \\ \mathbb{P}(R_{11} \setminus L_{11}) &= P(R_{11}) - 8 \binom{n}{2}^2 \left(\frac{1}{4}\right)^n + O(n^6 2^{-3n}). \end{aligned}$$

Proof. The expansion follows along the same reasoning as Proposition 7 using (14); in particular, with $t = 2 \binom{n}{2} 2^{-n}$ as before, we have

$$\mathbb{P}(R_{11} \setminus L_{11}) \geq t - G_1 - G_2.$$

\square

A similar analysis can be undertaken for D_{1111} ; we omit the details.

Lemma 2.

$$P(D_{11}) \geq 4 \binom{n}{2} \left(\frac{1}{2}\right)^n - \left(12 \binom{n}{2}^2 - 4 \binom{n}{2}\right) \left(\frac{1}{4}\right)^n.$$

$$P(D_{11}) = 4 \binom{n}{2} \left(\frac{1}{2}\right)^n - \left(12 \binom{n}{2}^2 - 4 \binom{n}{2} - 4 \binom{n}{3}\right) \left(\frac{1}{4}\right)^n + O(n^6 2^{-3n}),$$

$$\mathbb{P}(D_{1111}) = 2^4 \binom{n}{4} (3/8)^n + O(n^5 (3/16)^n),$$

$$\mathbb{P}(D_{1^6}) = 2^6 \binom{n}{6} \left(\frac{5}{16}\right)^n + O\left(n^7 \left(\frac{5}{32}\right)^n\right),$$

$$\mathbb{P}(D_{1^8}) = 2^8 \binom{n}{8} \left(\frac{35}{128}\right)^n + O\left(n^9 \left(\frac{35}{256}\right)^n\right),$$

$$\mathbb{P}(D_{21111}) = 2^5 \binom{n}{5} \binom{5}{1} \left(\frac{1}{4}\right)^n + O\left(n^6 \left(\frac{1}{8}\right)^n\right),$$

$$\mathbb{P}(D_{1^{10}}) = 2^{10} \binom{n}{10} \left(\frac{63}{256}\right)^n + O\left(n^{11} \left(\frac{63}{512}\right)^n\right),$$

$$\mathbb{P}(D_{21^6}) = 2^7 \binom{n}{7} \binom{7}{1} \left(\frac{60}{256}\right)^n + O\left(n^8 \left(\frac{60}{512}\right)^n\right),$$

$$\mathbb{P}(D_{1^{12}}) = 2^{12} \binom{n}{12} \left(\frac{231}{1024}\right)^n + O\left(n^{13} \left(\frac{231}{2048}\right)^n\right).$$

Proof. The first two equations follow from Proposition 7. The rest are proved similarly, but we omit the details. \square

Next we move to probabilities $\mathbb{P}(D_\lambda \cap D_\mu)$ for various choices of $\lambda \neq \mu$. Observe that the events involved can be highly positively correlated; for example, with $\lambda = 11, \mu = 1111$ we have $\mathbb{P}(D_\lambda \cap D_\mu)/(\mathbb{P}(D_\lambda) \times \mathbb{P}(D_\mu))$ grows exponentially fast, as $(4/3)^n$.

Proposition 9. *For two distinct novel partitions λ, μ , having j and k parts, respectively,*

$$\mathbb{P}(D_\lambda \cap D_\mu) \leq O\left(n^{k+j} (\max(r_\lambda, r_\mu)/2)^n\right).$$

The implicit constant in the big O varies with the choice of λ, μ .

Proof. Consider the event $R_\lambda \cap R_\mu = \bigcup \{vM = wM = 0\}$, where the union is over $v \in V_\lambda^{(n)}, w \in V_\mu^{(n)}$. The crucial ingredient is to show, with the notation of (6), that

$$(17) \quad \mathbb{P}(v \cdot X = w \cdot X = 0) \leq \max(r_\lambda, r_\mu)/2.$$

Without loss of generality, assume that the nonzero components of v are indexed by J , so $|J| = j$, and the nonzero components of w are indexed by K , with $|K| = k$. With $I = J \cup K$ having size $m = |I|$, the event of interest is based on m independent fair coins ϵ_i , $i \in I$, and can be expressed as

$$(18) \quad \left\{ \sum_{i \in J} v_i \epsilon_i = 0 \right\} \cap \left\{ \sum_{i \in K} w_i \epsilon_i = 0 \right\}.$$

Case 1: $J \neq K$. Without loss of generality, interchanging the λ and μ if needed, $I = \{1, 2, \dots, m\}$ and $m \in K \setminus J$. Condition on the values of the first $m-1$ coins, with a configuration that satisfies $\sum_{j \in J} v_j = 0$. These configurations belong to the event $vM = 0$, and hence have probabilities summing to at most r_λ . Each configuration, together with the requirement $wM = 0$, dictates the value needed for ϵ_m , which occurs with conditional probability $1/2$. The possible exchange of λ, μ at the start means that we have shown (17).

Case 2: $J = K$. Without loss of generality, rearranging the coordinates, and taking scalar multiples if needed, we can have $J = K = \{1, 2, \dots, k\}$ and $a := v_k = w_k \neq 0$. The event in (18) simplifies to

$$\left\{ -a\epsilon_k = \sum_{i=1}^{k-1} v_i \epsilon_i = \sum_{i=1}^{k-1} w_i \epsilon_i \right\}.$$

From this we conclude

$$\begin{aligned} r_\lambda &= \mathbb{P}(v \cdot X = 0) = \mathbb{P} \left(\sum_{i=1}^{k-1} v_i \epsilon_i \in \{\pm a\} \right) \\ &\geq 2 \mathbb{P} \left(\sum_{i=1}^{k-1} v_i \epsilon_i = \sum_{i=1}^{k-1} w_i \epsilon_i \in \{\pm a\} \right); \end{aligned}$$

inequality arises since the second sum, with weights w_i , might not even be in $\{\pm a\}$, and the factor of 2 arises since when the second sum is in the set, it dictates the choice of sign.

For the case where the potential null vectors are used on opposite sides, e.g., $L_\lambda \cap R_\mu$, we have

$$\mathbb{P}(L_\lambda \cap R_\mu) = O(\mathbb{P}(L_\lambda) \times \mathbb{P}(R_\mu)),$$

for the simple reason that conditioning of events of the form $Mw = 0$ with $w \in V_\mu^{(n)}$ only affects k of the columns, giving the bound above, with constant $(1/r_\mu)^k$ as the implicit constant for the big O. \square

Proposition 9 involves a bound that can have exponential decay as large as $(1/4)^n$. For the sake of proving Theorem 1, with error term involving $(7/32)^n$, we need a stronger bound, as given below.

Lemma 3. *For all novel partitions λ, μ , having j and k parts, respectively, with $\lambda \neq \mu$, and neither partition equal to the partition 11, we have,*

$$(19) \quad \mathbb{P}(D_{11} \cap D_{1111}) = 2^3 \binom{n}{4} \left(\frac{1}{4}\right)^n + O\left(n^5 \left(\frac{3}{16}\right)^n\right),$$

$$(20) \quad \mathbb{P}(D_{11} \cap D_\lambda) = O\left(n^{j+2} \left(\frac{3}{16}\right)^n\right),$$

$$(21) \quad \mathbb{P}(D_\lambda \cap D_\mu) = O\left(n^{k+j} \left(\frac{3}{16}\right)^n\right).$$

Proof. Equation (19) can be computed directly using inclusion exclusion, whereas Equations (20) and (21) use Proposition 9 with $\lambda = 1111$ since it is the most likely partition after 11. \square

Finally, we note a trivial lemma to simplify the coefficient for 4^{-n} in the expansion of Theorem 1,

Lemma 4.

$$(22) \quad \frac{1}{2} \left(\binom{n}{2}^2 - \binom{n}{2} \right) = 3 \binom{n}{4} + 3 \binom{n}{3}.$$

Proof. Either simplify algebraically or note that the left hand side is the number of ways to choose any two unordered distinct pairs of unordered distinct pairs of n objects. The right hand side counts the number of ways to select these pairs where all four indices are distinct and can be placed in 3 distinct configurations, and the second term counts the number of pairs that share a common index, of which there are 3 choices for the repeated index. \square

5. INTERACTION OF LEFT AND RIGHT NULL VECTORS

Proposition 8 gives a lower bound on $\mathbb{P}(R_{11} \setminus L_{11}) = \mathbb{P}(L_{11} \setminus R_{11})$ which has, as a corollary,

$$\mathbb{P}(S \setminus L_{11}) \geq \mathbb{P}(R_{11} \setminus L_{11}) \sim \mathbb{P}(R_{11}) = \mathbb{P}(L_{11}),$$

and omitting the middle terms, and writing $a_n \gtrsim c_n$ to mean that there exists b_n with $a_n \geq b_n$ and $b_n \sim c_n$, we have

$$(23) \quad \mathbb{P}(S \setminus L_{11}) \gtrsim \mathbb{P}(L_{11}).$$

Expressing (23) in terms of left null vectors, with the outer union on the left taken over all novel partitions λ of length less than or equal to n , other than 11, we have

$$\mathbb{P} \left(\bigcup_{\lambda \neq 11} \bigcup_{v \in V_\lambda^{(n)}} \{vM = 0\} \right) \gtrsim \mathbb{P} \left(\bigcup_{v \in V_{11}^{(n)}} \{vM = 0\} \right).$$

Writing this with L_λ for the event that M has a left null vector of template λ , the above display can be rewritten as

$$\sum_{\lambda \neq 11} \mathbb{P}(L_\lambda) \gtrsim \mathbb{P}(L_{11}).$$

We believe that to prove sharp upper bounds on P_n , say as given by (1) or (4), it will be necessary to consider the effect of conditioning on D_{11}^c . Propositions 10 and 11 might be a first step in this direction.

Proposition 10. *Suppose that λ is a novel partition of length k , with $k = n$. Let $2p = |\lambda^{\perp B}|$. Recall that R_{11} is the event that our n by n matrix M has a right null vector of the form $e_i \pm e_j$. For every $v \in V_\lambda$,*

$$\frac{\mathbb{P}(vM = 0 | R_{11}^c)}{\mathbb{P}(vM = 0)} = \frac{(p)_n}{p^n}.$$

Proof. The hypothesis $k = n$ is essential: if x denotes a column of M , then, thanks to $k = n$, we know that $x \in v^{\perp B}$. There are p choices for the “direction” $\{-x, x\}$ with $x \in v^{\perp B}$, and different columns of M must choose different directions, otherwise the event R_{11} would occur. By giving the ratio of the conditional probability to the unconditional probability, factors of 2, for choosing between x and $-x$, for each column, cancel. \square

Proposition 11. *Suppose that λ is a novel partition of length k , with $k = n - 1$. Let $2p = |\lambda^{\perp B}|$. Recall that R_{1111} is the event that our n by n matrix M has a right null vector of the form $e_{j_1} \pm e_{j_2} \pm e_{j_3} \pm e_{j_4}$. For every $v \in V_\lambda^{(n)}$, as specified by Definition 2,*

$$\frac{\mathbb{P}(vM = 0 | (R_{11} \cup R_{1111})^c)}{\mathbb{P}(vM = 0)} = \frac{(p)_n}{p^n} + \frac{1}{2p} \binom{n}{2} \frac{(p)_{n-1}}{p^{n-1}}.$$

Proof. The hypothesis $k = n - 1$ is essential. Without loss of generality, assume that $v_n = 0$, so $v = (w, 0)$ with $w \in V_\lambda$. Let $x = (y, s)$ denote a column of M , where y gives the first $n - 1$ coordinates, and $s \in \{-1, +1\}$. Then, thanks to $k = n - 1$ and $v_n = 0$, we know that $y \in w^{\perp B}$. There are p choices for the “direction” $\{-y, y\}$ — restricting to the first $n - 1$ coordinates, with $y \in w^{\perp B}$, and different columns of M must choose different directions, apart from possibly one pair of columns, where the columns in a pair may share the underlying $n - 1$ direction, but have opposite choices of s for their n th coordinate. (If three columns share the underlying $n - 1$ direction, the event R_{11} would occur; if two pairs of columns share, then event R_{1111} would occur.) \square

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